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A Zero-Truncated Two-Parameter Poisson-Lindley Distribution with an Application to Biological Science

Sıfır-Kesilmiş İki Parametreli Poisson-Lindley Dağılımı ve Biyoloji Biliminde Uygulaması

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ABSTRACT In this paper, a zero-truncated two-parameter Poisson-Lindley distribution (ZTTPPLD) which includes zero-truncated Poisson-Lindley distribution (ZTPLD) as a particular case has been obtained by compounding size-biased Poisson distribution (SBPD) with an assumed continuous distribution. Its raw moments and central moments have been given. The coefficients of variation, skewness, kurtosis, and index of dispersion have been obtained and their nature and behavior have been discussed. The estimation of parameters has been discussed using both method of moments and maximum likelihood estimation. The goodness of fit of ZTTPPLD has been discussed with an example from biological science and the fit has been found better as compared with zero-truncated Poisson distribution (ZTPD) and zero- truncated Poisson- Lindley distribution (ZTPLD).

Keywords: Zero-truncated distribution; two-parameter poisson-lindley distribution; moments; maximum likelihood estimation; goodness of fit

ÖZET Bu makalede, varsayılan bir sürekli dağılım ile size-biased Poisson dağılımı birleştirilerek, özel bir durum olarak sıfır-kesilmiş Poisson-Lindley dağılımı (ZTPLD) içeren bir sıfır-kesilmiş iki-parametreli Poisson-Lindley dağılımı (ZTTPPLD) elde edilmiştir. Ham momentler ve merkezi momentler verilmiştir. Değişim katsayıları, çarpıklık, basıklık ve yayılım indeksi elde edilmiş; doğaları ve davranışları tartışılmıştır. Parametrelerin tahmini hem momentler yöntemi hem de en çok olabilirlik tahmini kullanılarak ele alınmıştır. ZTTPPLD'nin uyum iyiliği biyoloji biliminden bir örnek ile tartışılmıştır. Uyum, sıfır-kesilmiş Poisson dağılımına (ZTPD) ve sıfır-kesilmiş Poisson Lindley dağılımına (ZTPLD) göre daha iyi bulunmuştur.

Anahtar Kelimeler: Sıfır-kesilmiş dağılım; iki-parametreli poisson-lindley dağılımı; momentler; en çok olabilirlik tahmini; uyum iyiliği

In probability theory, zero-truncated distributions are certain discrete distributions having support the set of positive integers. Zero-truncated distributions are suitable models for modeling data when the data to be modeled originate from a mechanism which generates data excluding zero counts.

Suppose $P_0(x;\theta)$ is the original distribution. Then the zero-truncated version of $P_0(x;\theta)$ can be defined as

$$P_1(x;\theta)=\frac{P_0(x;\theta)}{1-P_0(0;\theta)} \quad ;x=1,2,3,... \tag{1.1}$$

Shanker and Mishra (2014) has obtained a two-parameter Poisson-Lindley distribution (TPPLD) defined by its pmf[7]

$$P_0(x;\theta,\alpha)=\frac{\theta^2}{\theta\alpha+1}\frac{x+\alpha(\theta+1)+1}{(\theta+1)^{x+2}};x=0,1,2,...,\theta>0,\theta\alpha>-1 \tag{1.2}$$

It can be easily verified that at $\alpha=1$, TPPLD (1.2) reduces to the Poisson-Lindley distribution (PLD) introduced by Sankaran (1970) having pmf[5]

$$P_2(x;\theta)=\frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}\;;x=0,1,2,...,\theta>0 \tag{1.3}$$

Shanker and Mishra (2014) have studies the mathematical and statistical properties, estimation of parameters of TPPLD and its applications to model counts data [7]. It should be noted that PLD is also a Poisson mixture of Lindley distribution, introduced by Lindley (1958) [4]. Shanker and Hagos (2015) have discussed the applications of PLD for modeling data from biological sciences and concluded that PLD gives much closer fit than Poisson distribution[8].

The TPPLD (1.2) is a Poisson mixture of two-parameter Lindley distribution (TPLD) of Shanker and Mishra (2013) having pdf [6]

$$f_1(x;\theta,\alpha)=\frac{\theta^2}{\theta\alpha+1}(\alpha+x)e^{-\theta x};x>0,\theta>0,\theta\alpha>-1 \tag{1.4}$$

The Lindley distribution, introduced by Lindley (1958) and studied in detail by Ghitany et al (2008 b) is a particular case of (1.4) for $\alpha=1$ [2,4]. Shanker et al (2016) have discussed various interesting properties including moment generating function, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measures and stress-strength reliability and applications of TPLD to model lifetime data from biomedical science and engineering[10].

In this paper, a zero-truncated two-parameter Poisson-Lindley distribution (ZTTPPLD), of which zero-truncated Poisson-Lindley distribution (ZTPLD) is a particular case, has been obtained by compounding size-biased Poisson distribution (SBPD) with a continuous distribution. Its raw moments and central moments have been obtained. The expressions for coefficient of variation, skewness, kurtosis, and index of dispersion have been obtained and their nature and behavior have been discussed graphically. Maximum likelihood estimation has been discussed for estimating the parameters of ZTTPPLD and its goodness of fit has been compared with ZTPLD and ZTPD with an example from biological science.

ZERO-TRUNCATED TWO-PARAMETER POISSON-LINDLEY DISTRIBUTION

Using (1.1) and (1.2), the pmf of zero-truncated two-parameter Poisson-Lindley distribution (ZTTPPLD) can be obtained as

$$P_3(x;\theta,\alpha)=\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\frac{x+\alpha(\theta+1)+1}{(\theta+1)^x}\;;x=1,2,3,...,\theta>0,\theta^2\alpha+\theta\alpha+2\theta+1>0 \tag{2.1}$$

It can be easily verified that at $\alpha=1$, (2.1) reduces to the pmf of zero-truncated Poisson-Lindley distribution (ZTPLD) introduced by Ghitany *et al* (2008) and defined by.[3]

$$P_4(x;\theta)=\frac{\theta^2}{\theta^2+3\theta+1}\frac{x+\theta+2}{(\theta+1)^x}\;;x=1,2,3,....,\theta>0 \tag{2.2}$$

Shanker *et al* (2015) have done extensive study on the comparison between ZTPD and ZTPLD with respect to their applications in data-sets excluding zero-counts and showed that in demography and biological sciences ZTPLD gives better fit than ZTPD while in social sciences ZTPD gives better fit than ZTPLD[9].

The p.m.f. of zero-truncated Poisson-distribution (ZTPD) is given by

$$P_5(x;\theta)=\frac{e^{-\theta}\,\theta^x}{\left(1-e^{-\theta}\right)x!}\quad ;x=1,2,3,....,\theta>0 \tag{2.3}$$

The nature and behavior of ZTTPPLD for varying values of parameters θ and α are explained graphically in fig.1. It is obvious from the graphs of the pmf of ZTTPPLD that it is monotonically decreasing for increasing values of the parameters θ and α. If θ is constant and α increases, the graph of the pmf of ZTTPPLD shift upward slowly and decreasing slowly for increasing value of x. If α is constant and θ increases, the graph of the pmf of ZTTPPLD starts from higher value and decreasing fast with increasing value of x. That is the change in the parameter θ makes a difference in the shape of the pmf of ZTTPPLD and thus the parameter θ is the dominating parameter. The change in the value of parameter α does not make much impact on the shape of the pmf of ZTTPPLD.

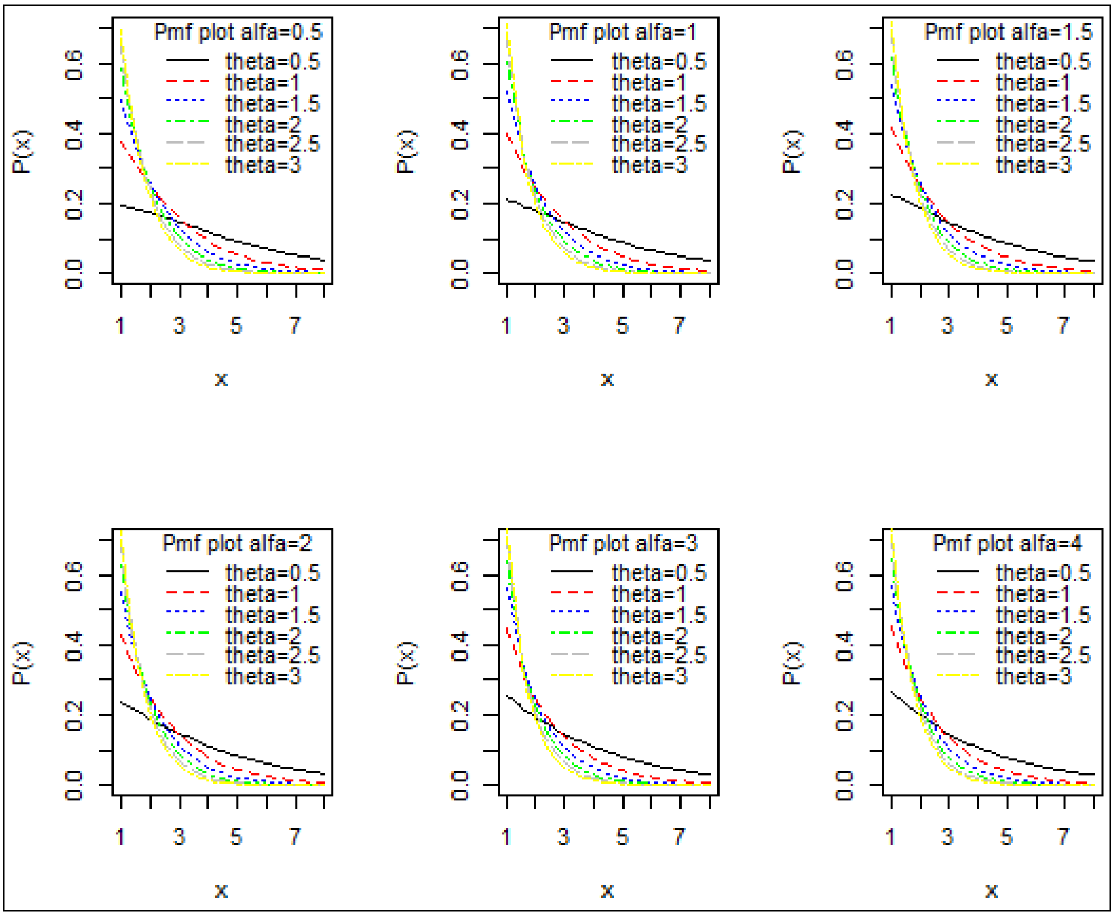

FIGURE 1. Graph of the probability mass function of ZTTPPLD for varying values of parameters θ and α

It should be noted that it is very tedious and complicated to find the moments of ZTTPPLD (2.1) directly. To find moments of ZTTPPLD in an easy and interesting way, firstly we have obtained the pmf of ZTTPPLD as a size-biased Poisson mixture of an assumed continuous distribution.

The ZTTPPLD (2.1) can also be obtained from size-biased Poisson distribution (SBPD) having pmf

$$g(x\mid\lambda)=\frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\quad ;x=1,2,3,\ldots,\lambda>0 \tag{2.4}$$

when the parameter λ of SBPD follows a continuous distribution having p.d.f.

$$h(\lambda;\theta,\alpha)=\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda};\lambda>0,\theta>0,\theta^2\alpha+\theta\alpha+2\theta+1>0 \tag{2.5}$$

Since the parameter $\lambda>0$ of SBPD, we assumed its distribution to be a continuous distribution (2.5) so that the mixture of SBPD (2.4) with (2.5) gives ZTTPPLD (2.1) and thus will be useful for obtaining moments of the proposed distribution.

Thus, the pmf of ZTTPPLD can be obtained as

$$P(x;\theta,\alpha)=\int_0^\infty g(x\mid\lambda)\cdot h(\lambda;\theta,\alpha)\,d\lambda$$

$$=\int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\cdot\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda}\,d\lambda \tag{2.6}$$

$$=\frac{\theta^2}{\left[\theta^2\alpha+\theta\alpha+2\theta+1\right]\Gamma(x)}\int_0^\infty e^{-(\theta+1)\lambda}\left[(\theta+1)\lambda^x+\left\{\alpha(\theta+1)+1\right\}\lambda^{x-1}\right]d\lambda$$

$$=\frac{\theta^2}{\left[\theta^2\alpha+\theta\alpha+2\theta+1\right]\Gamma(x)}\left[\frac{(\theta+1)\Gamma(x+1)}{(\theta+1)^{x+1}}+\frac{\left\{\alpha(\theta+1)+1\right\}\Gamma(x)}{(\theta+1)^x}\right]$$

$$=\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\frac{x+\alpha(\theta+1)+1}{(\theta+1)^x};x=1,2,3,\ldots,\ \theta>0,\theta^2\alpha+\theta\alpha+2\theta+1>0$$

which is the p.m.f. of ZTTPPLD with parameter θ and α as given in (2.1).

MOMENTS AND RELATED MEASURES OF ZTTPPLD

The r th factorial moment about origin of ZTTPPLD (2.1) can be obtained as

$\mu_{(r)}'=E\left[E\left(X^{(r)}\mid\lambda\right)\right]$; where $X^{(r)}=X(X-1)(X-2)\ldots.(X-r+1)\cdot$

Using (2.6), we have

$$\mu_{(r)}'=\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\int_0^\infty\left[\sum_{x=1}^\infty x^{(r)}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right]\cdot\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\int_0^\infty\left[\lambda^{r-1}\sum_{x=r}^\infty x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\cdot\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda}d\lambda$$

Taking $y = x - r$, we get

$$\mu_{(r)}' = \frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\int_0^\infty \left[\lambda^{r-1}\sum_{y=0}^{\infty}(y+r)\frac{e^{-\lambda}\lambda^y}{y!}\right]\cdot\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^2}{\theta^2\alpha+\theta\alpha+2\theta+1}\int_0^\infty \lambda^{r-1}(\lambda+r)\cdot\left[(\theta+1)\lambda+\alpha(\theta+1)+1\right]e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get the expression for the r th factorial moment about origin of ZTTPPLD (2.1) as

$$\mu_{(r)}' = \frac{r!(\theta+1)^2(\theta\alpha+r+1)}{\theta^r\left[\theta^2\alpha+\theta\alpha+2\theta+1\right]}\;;r=1,2,3,... \tag{3.1}$$

Substituting $r = 1, 2, 3, \text{and } 4$ in equation (3.1), the first four factorial moments about origin can be obtained and using the relationship between moments about origin and factorial moments about origin, the first four moments about origin of ZTTPPLD (2.1) are obtained as

$$\mu_1' = \frac{(\theta+1)^2(\theta\alpha+2)}{\theta\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)}$$

$$\mu_2' = \frac{(\theta+1)^2\left\{\theta\alpha(\theta+2)+2(\theta+3)\right\}}{\theta^2\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)}$$

$$\mu_3' = \frac{(\theta+1)^2\left\{\theta\alpha\left(\theta^2+6\theta+6\right)+2\left(\theta^2+9\theta+12\right)\right\}}{\theta^3\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)}$$

$$\mu_4' = \frac{(\theta+1)^2\left\{\theta\alpha\left(\theta^3+14\theta^2+36\theta+24\right)+2\left(\theta^3+21\theta^2+72\theta+60\right)\right\}}{\theta^4\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)}$$

Again using the relationship between moments about origin and moments about mean, the moments about mean of ZTTPPLD (2.1) are obtained as

$$\mu_2 = \sigma^2 = \frac{(\theta+1)^2\left\{\theta^3\alpha^2+(\alpha+5)\theta^2\alpha+(4\alpha+6)\theta+2\right\}}{\theta^2\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)^2}$$

$$\mu_3 = \frac{(\theta+1)^2\left\{\begin{array}{l}\theta^6\alpha^3+(4\alpha+7)\theta^5\alpha^2+\left(5\alpha^2+28\alpha+16\right)\theta^4\alpha+\left(2\alpha^3+33\alpha^2+59\alpha+12\right)\theta^3\\+\left(12\alpha^2+54\alpha+38\right)\theta^2+(12\alpha+22)\theta+4\end{array}\right\}}{\theta^3\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)^3}$$

$$\mu_4 = \frac{(\theta+1)^2\left\{\begin{array}{l}\theta^9\alpha^4+(12\alpha+9)\theta^8\alpha^3+\left(39\alpha^2+114\alpha+30\right)\theta^7\alpha^2+\left(55\alpha^3+363\alpha^2+389\alpha+44\right)\theta^6\alpha\\+\left(36\alpha^4+492\alpha^3+1147\alpha^2+572\alpha+24\right)\theta^5+\left(9\alpha^4+306\alpha^3+1376\alpha^2+1497\alpha+308\right)\theta^4\\+\left(72\alpha^3+720\alpha^2+1508\alpha+686\right)\theta^3+\left(132\alpha^2+636\alpha+554\right)\theta^2+(96\alpha+192)\theta+24\end{array}\right\}}{\theta^4\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)^4}$$

Finally, the coefficient of variation (C.V), coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of ZTTPPLD (2.1) are obtained as

$$C.V.=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\left\{\theta^3\alpha^2+(\alpha+5)\theta^2\alpha+(4\alpha+6)\theta+2\right\}}}{(\theta+1)(\theta\alpha+2)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{(\mu_2)^{3/2}}=\frac{\left\{\begin{array}{l}\theta^6\alpha^3+(4\alpha+7)\theta^5\alpha^2+\left(5\alpha^2+28\alpha+16\right)\theta^4\alpha+\left(2\alpha^3+33\alpha^2+59\alpha+12\right)\theta^3\\+\left(12\alpha^2+54\alpha+38\right)\theta^2+(12\alpha+22)\theta+4\end{array}\right\}}{(\theta+1)\left\{\theta^3\alpha^2+(\alpha+5)\theta^2\alpha+(4\alpha+6)\theta+2\right\}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^{\,2}}=\frac{\left\{\begin{array}{l}\theta^9\alpha^4+(12\alpha+9)\theta^8\alpha^3+\left(39\alpha^2+114\alpha+30\right)\theta^7\alpha^2+\left(55\alpha^3+363\alpha^2+389\alpha+44\right)\theta^6\alpha\\+\left(36\alpha^4+492\alpha^3+1147\alpha^2+572\alpha+24\right)\theta^5+\left(9\alpha^4+306\alpha^3+1376\alpha^2+1497\alpha+308\right)\theta^4\\+\left(72\alpha^3+720\alpha^2+1508\alpha+686\right)\theta^3+\left(132\alpha^2+636\alpha+554\right)\theta^2+(96\alpha+192)\theta+24\end{array}\right\}}{(\theta+1)^2\left\{\theta^3\alpha^2+(\alpha+5)\theta^2\alpha+(4\alpha+6)\theta+2\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\theta^3\alpha^2+(\alpha+5)\theta^2\alpha+(4\alpha+6)\theta+2}{\theta(\theta\alpha+2)\left(\theta^2\alpha+\theta\alpha+2\theta+1\right)}$$

The nature of coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion of ZTTPPLD (2.1) are shown graphically in Figure 2. From the graphs in Figure 2, it is clear that C.V and index of dispersion of ZTTPPLD are decreasing for increasing values of the parameters while coefficients of Skewness and Kurtosis are increasing for increasing values of parameters.

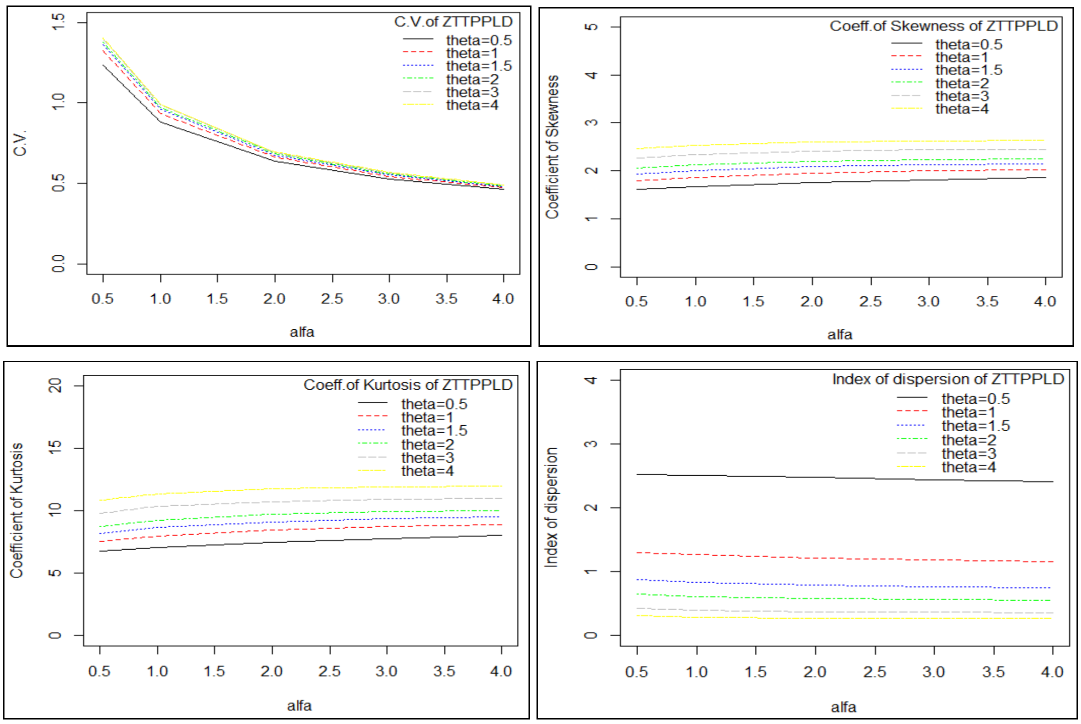

FIGURE 2. Coefficient of variation (C.V.), Coefficient of skewness, coefficient of kurtosis and index of dispersion plot for different values of α and θ

ESTIMATION OF PARAMETERS

METHOD OF MOMENT ESTIMATES (MOME) OF PARAMETERS

Since ZTTPPLD (2.1) has two parameters θ and α to be estimated, the method of moment is based on equating the ratio of the first two probabilities and the mean. From (2.1), we have

$$p_1=\frac{\theta^2(\theta\alpha+\alpha+2)}{(\theta+1)(\theta^2\alpha+\theta\alpha+2\theta+1)} \quad \text{and} \quad p_2=\frac{\theta^2(\theta\alpha+\alpha+3)}{(\theta+1)^2(\theta^2\alpha+\theta\alpha+2\theta+1)}.$$

Taking the ratio of these probabilities, we get

$$\frac{p_2}{p_1}=\frac{(\theta\alpha+\alpha+3)}{(\theta+1)(\theta\alpha+\alpha+2)}$$

After a little algebraic simplification, above equation reduces to

$$\alpha(\theta+1)=\frac{3p_1-2(\theta+1)p_2}{(\theta+1)p_2-p_1} \tag{4.1.1}$$

Again, simplifying the expression of the mean of ZTTPPLD (2.1), we get

$$\alpha(\theta+1)=\frac{2(1-\mu)\theta^2+(4-\mu)\theta+2}{\theta(\mu\theta-\theta-1)} \tag{4.1.2}$$

Now, equating equation (4.1.1) and (4.1.2), we get a quadratic equation in θ as

$$A\theta^2+B\theta+C=0 \tag{4.1.3}$$

where

$$\left.\begin{aligned} A&=(1-\mu)p_1+(2-\mu)p_2\\ B&=(\mu-1)p_1+(4-\mu)p_2\\ C&=2(p_1-p_2)\end{aligned}\right\} \tag{4.1.4}$$

From equation (4.1.3) , MOME $\tilde{\theta}$ of θ can be obtained from

$$\tilde{\theta}=\frac{-B+\sqrt{B^2-4AC}}{2A} \tag{4.1.5}$$

Replacing the population mean μ by the corresponding sample mean $\bar{x}$ and p_1 and p_2 from the respective probabilities from the given data set, values of A, B, and C in (4.1.4) can be obtained and substituting these values in (4.1.5) will give MOME estimate $\tilde{\theta}$ of θ. After obtaining the MOME $\tilde{\theta}$, MOME $\tilde{\alpha}$ of α can be obtained either from

$$\tilde{\alpha}=\frac{1}{\theta+1}\left[\frac{3p_1-2(\theta+1)p_2}{(\theta+1)p_2-p_1}\right] \tag{4.1.6}$$

Or

$$\tilde{\alpha}=\frac{1}{\theta+1}\left[\frac{2(1-\mu)\theta^2+(4-\mu)\theta+2}{\theta(\mu\theta-\theta-1)}\right] \tag{4.1.7}$$

MAXIMUM LIKELIHOOD ESTIMATES (MLE) OF PARAMETERS

Let $(x_1, x_2, ..., x_n)$ be a random sample of size n from the ZTTPPLD (2.1) and let f_x be the observed frequency in the sample corresponding to $X = x \ (x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the ZTTPPLD (2.1) is given by

$$L = \left(\frac{\theta^2}{\theta^2 \alpha + \theta \alpha + 2\theta + 1} \right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}} \prod_{x=1}^{k} \left[x + \alpha(\theta+1) + 1 \right]^{f_x}$$

The log likelihood function is thus obtained as

$$\log L = n \log \left(\frac{\theta^2}{\theta^2 \alpha + \theta \alpha + 2\theta + 1} \right) - \sum_{x=1}^{k} x f_x \log(\theta+1) + \sum_{x=1}^{k} f_x \log\left[x + \alpha(\theta+1) + 1 \right]$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of ZTTPPLD (2.1) is the solutions of the following log likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n(2\theta \alpha + \alpha + 2)}{\theta^2 \alpha + \theta \alpha + 2\theta + 1} - \frac{n \bar{x}}{\theta + 1} + \sum_{x=1}^{k} \frac{\alpha f_x}{\left[x + \alpha(\theta+1) + 1 \right]} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{-n(\theta^2 + \theta)}{\theta^2 \alpha + \theta \alpha + 2\theta + 1} + \sum_{x=1}^{k} \frac{(\theta+1) f_x}{\left[x + \alpha(\theta+1) + 1 \right]} = 0$$

where $\bar{x}$ is the sample mean.

These two log likelihood equations do not seem to be solved directly because MLE's of parameters of ZTTPPLD do not have closed form. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n\left\{ \alpha^2 (2\theta^2 + 2\theta + 1) + 4\theta\alpha + 2(\alpha + 2) \right\}}{\left[\theta^2 \alpha + \theta \alpha + 2\theta + 1 \right]^2} + \frac{n\bar{x}}{(\theta+1)^2} - \sum_{x=1}^{k} \frac{\alpha^2 f_x}{\left[x + \alpha(\theta+1) + 1 \right]^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n(\theta^2 + \theta)^2}{(\theta^2 \alpha + \theta \alpha + 2\theta + 1)^2} - \sum_{x=1}^{k} \frac{(\theta+1)^2 f_x}{\left[x + \alpha(\theta+1) + 1 \right]^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{-n(2\theta^2 + 2\theta + 1)}{(\theta^2 \alpha + \theta \alpha + 2\theta + 1)^2} + \sum_{x=1}^{k} \frac{(x+1) f_x}{\left[x + \alpha(\theta+1) + 1 \right]^2} = \frac{\partial^2 \log L}{\partial \alpha \partial \theta}$$

For the maximum likelihood estimates $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of ZTTPPLD (2.1), following equations can be solved

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively. Note that the initial values (θ_0, α_0) are the values given by method of moments. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

GOODNESS OF FIT

The goodness of fit of ZTTPPLD has been discussed with a count data and the fit has been compared with ZTPD and ZTPLD. In this section, the goodness of fit of ZTPD, ZTPLD, and ZTTPPLD have been compared for a data set due to Finney and Varley (1955) who gave counts of number of flower having number of fly eggs [1]. The values of ML estimate of parameters, standard error of parameters, chi-square (c^2), p-values, $-2\log L$ and AIC (Akaike information criterion) are given in the table1. From the table it is obvious that the values of chi-square and AIC are minimum for ZTTPPLD as compared to ZTPLD and ZTPD, so ZTTPPLD can be considered as an important model for biological science data.

TABLE 1: Number of flower heads with number of fly eggs, reported by Finney and Varley (1955).[1]

Number of fly eggs	Number of flowers	Expected Frequency		
		ZTPD	ZTPLD	ZTTPPLD
1	22	15.3	26.8	20.8
2	18	21.8	19.8	22.2
3	18	20.8	14.0	16.9
4	11	14.9	9.5	11.3
5	9	8.5	6.3	7.0
6	6	4.0	4.2	4.2
7	3	1.7	2.7	2.4
8	0	0.6	1.7	1.4
9	1	0.4	3.0	1.8
Total	88	88.0	88.0	88.0
ML Estimate		$\hat{\theta} = 2.8604$	$\hat{\theta} = 0.7186$	$\hat{\theta} = 1.02088$ $\hat{\alpha} = -0.56444$
Standard Error		0.1926	0.0759	$S.E(\hat{\theta}) = 0.1681$ $S.E(\hat{\alpha}) = 0.1832$
c^2		6.648	3.780	1.518
d.f.		4	4	3
P-value		0.1557	0.4366	0.6781
$-2\log L$		333.09	334.76	330.48
AIC		336.76	335.09	334.48

The AIC has been computed using $AIC = -2\log L + 2k$, where k is the number of parameters involved in the respective distributions.

The corresponding fitted pmf of ZTPD, ZTPLD, and ZTTPPLD along with the original data points for the data set in table 1 are given in figure 3.

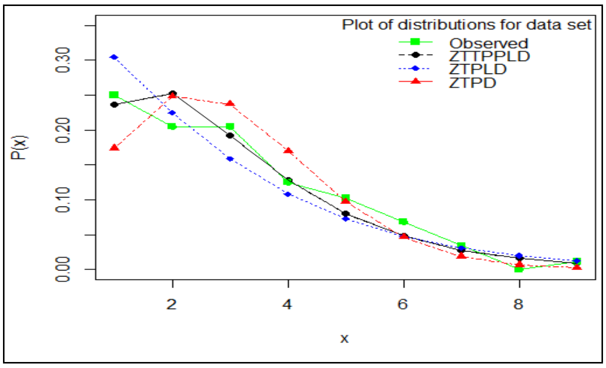

FIGURE 3. Corresponding fitted pmf of ZTPD, ZTPLD and ZTTPLD for data set in Table 1.

CONCLUDING REMARKS

In this paper, a zero-truncated two-parameter Poisson-Lindley distribution (ZTTPPLD), of which zero-truncated Poisson-Lindley distribution (ZTPLD) is a particular case, has been obtained by compounding size-biased Poisson distribution (SBPD) with a continuous distribution. Its moments, and moments based measures including coefficient of variation, skewness, kurtosis, and index of dispersion have been obtained and their nature and behavior have been studied graphically. Method of maximum likelihood estimation has been discussed for estimating the parameters of ZTTPPLD and the goodness of fit has been discussed with a data from biological science and the fit shows quite satisfactory fit over ZTPD and ZTPLD. The nature of probability mass function of fitted distributions for the data set has also been discussed graphically.

Conflict of Interest

Authors declared no conflict of interest or financial support.

Authorship Contributions

Idea/Concept: *Rama Shanker*

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Data Collection and/or Processing: *Kamlesh Kumar Shukla*

Analysis and/or Interpretation: *Rama Shanker*

Literature Review: *Rama Shanker*

Writing the Article: *Rama Shanker*

Critical Review: *Kamlesh Kumar Shukla*

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